

University of Groningen

Lagrangian modeling of switching electrical networks

Scherpen, Jacquélien M.A.; Jeltsema, Dimitri; Klaassens, J. Ben

Published in:
Systems & Control Letters

DOI:
[10.1016/S0167-6911\(02\)00290-6](https://doi.org/10.1016/S0167-6911(02)00290-6)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2003

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Scherpen, J. M. A., Jeltsema, D., & Klaassens, J. B. (2003). Lagrangian modeling of switching electrical networks. *Systems & Control Letters*, 48(5), 365-374. [https://doi.org/10.1016/S0167-6911\(02\)00290-6](https://doi.org/10.1016/S0167-6911(02)00290-6)

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Lagrangian modeling of switching electrical networks

Jacqueline M.A. Scherpen*, Dimitri Jeltsema, J. Ben Klaassens

*Control Systems Engineering, Faculty of Information Technology and Systems, Delft University of Technology,
P.O. Box 5031, 2600 GA Delft, Netherlands*

Received 13 July 2001; received in revised form 3 September 2002; accepted 18 September 2002

Abstract

In this paper, a general and systematic method is presented to model topologically complete electrical networks, with or without multiple or single switches, within the Euler–Lagrange framework. Apart from the physical insight that can be obtained in this way, the framework has proven to be useful for the application of passivity-based control techniques, which on a case by case basis already has shown to be useful for the control of power converters within the class of switching electrical networks. The switches are assumed to be ideal, and pulse-width modulation is taken into account. Magnetic coupling of inductive elements is also included in the framework.

© 2002 Elsevier Science B.V. All rights reserved.

Keywords: Euler–Lagrange equations; Passivity; Energy; Switching electrical networks; Power converters

1. Introduction

During the last decades modeling, design and control techniques for switched-mode power converters have obtained a lot of attention. Power converters play a primary role in modern power systems for many applications, but in the applications so far, the controller design is always based on the linearized models. In recent developments, the power converters are considered from an energy storage modeling point of view, with the prime objective to design a passivity-based controller (PBC), such that the non-linearities of the models can be taken into account, and the passivity properties are explored. It is shown in [9,12,14]

that the conventional average pulse-width modulation (PWM) models of the classical Buck, Boost, Buck–Boost converters correspond to systems derived from classical Euler–Lagrange (or Hamiltonian [5]) dynamic considerations. The approach consists of establishing a suitable set of average *Euler–Lagrange (EL) parameters* modulated by the duty-ratio function. A main advantage of underscoring the physical properties in terms of energy storage and power flow for switching power converters by modeling via the EL or Hamiltonian framework is that these properties can be exploited at the feedback controller design stage. In particular, PBC design strongly relies on the explicit presence of energy storage in the structure of the dynamics [9], and the EL modeling framework offers a useful tool for that.

So far, the physical EL and Hamiltonian models were given for a limited set of single switch converter structures, based on case by case studies of the Boost, Buck and Buck–Boost converters in [9,14] and of the

* Corresponding author. Tel.: +31152786152; fax: +311527-86679.

E-mail address: j.m.a.scherpen@its.tudelft.nl
(J.M.A. Scherpen).

Ćuk converter in [12]. In this paper, we develop a procedure that results in the EL parameters for a broad class, the so-called topologically complete class [18], of (multi, single, or none) switching electrical network structures, where the EL parameters are extended with constraint equations stemming from Kirchhoff's current laws and where magnetic coupling can be included. We assume switches to be ON and OFF, but we will also see that this approach works for the PWM models of the networks. A main advantage of the proposed approach is the transparent structure, stemming from the fact that the EL equations are easily obtained from the energy in the system, while they essentially represent the Kirchhoff voltage laws, and the constraint equations represent the Kirchhoff current laws.

Though our prime objective is to build models for the application of PBC design to switch mode electrical circuits, the modeling procedure is also applicable to electrical circuits without switches. That observation may give rise to consider e.g. bond-graph modeling as an alternative to the proposed modeling procedure. However, though the use of energy storage and power flow are basic ingredients in bond-graphs, EL and Hamiltonian modeling, and the relation between these modeling frameworks is known, see e.g. [15], PBC design is most easily and straightforwardly applied in terms of the EL and Hamiltonian frameworks. Here, we focus on the clear structure offered by the EL framework, which has its own specific advantages for the application of PBC, see [9]. It should also be noted that all modeling strategies, including strategies like the classical mesh current and node voltage analysis result in the same dynamical behavior provided that the modeled lumps are the same.

In Section 2 we present the general procedure to develop an EL model for electrical networks with or without (single or multiple) switches. The procedure is illustrated with three examples, a network without switches, a Ćuk converter circuit, and a three-phase boost rectifier circuit. In the three-phase boost rectifier circuit some other issues, related to minimality of the obtained model will also be studied. Then, in Section 3 we include coupled magnetics in our Lagrangian modeling framework, i.e., the possible occurrence of magnetic coupling between inductive elements. This is illustrated with the example of a magnetically coupled Ćuk converter circuit. Finally, in Section 4 we end with the conclusions.

2. EL modeling of (switching) networks

The EL dynamics of electric circuits can be classically characterized by the following set of non-linear differential equations:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}) \right) - \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) = - \frac{\partial \mathcal{D}}{\partial \dot{q}}(\dot{q}) + \mathcal{F}_q, \quad (1)$$

where \dot{q} is the vector of flowing current and q is the vector of its time integral, i.e. the electric charge, see [9,14]. The vector of electric charge constitutes the generalized coordinates describing the circuit. This vector is assumed to have n components, represented by q_1, \dots, q_n . \mathcal{L} is the Lagrangian of the system, defined as the difference between the magnetic co-energy of the circuit, denoted by $\mathcal{T}(q, \dot{q})$, and the electric field energy of the circuit, denoted by $\mathcal{V}(q)$, i.e.

$$\mathcal{L}(q, \dot{q}) = \mathcal{T}(q, \dot{q}) - \mathcal{V}(q). \quad (2)$$

\mathcal{T} is given by the sum of the magnetic co-energies of the inductive elements in terms of the currents through the inductors, while \mathcal{V} is given by the sum of the electric field energies of the capacitive elements in terms of the charges on the capacitors. The function $\mathcal{D}(\dot{q})$ is the Rayleigh dissipation function¹ of the system, and represents a measure of the free energy (or power) that is lost through dissipation, either through losses in the dynamic elements or through the load that is modeled as a dissipative element. The vector $\mathcal{F}_q = (\mathcal{F}_{q_1}, \dots, \mathcal{F}_{q_n})^T$ represents the ordered components of the set of generalized forcing functions, or voltage sources, associated with each generalized coordinate. The EL equations (1) represent a generalized force balance, or in other words, represent an effort variable balance. In the electrical domain this means that the equations constitute a voltage balance that corresponds with the Kirchhoff voltage laws, where the branch relations are already substituted into the equation. The choice of canonical coordinates charge and current, and the corresponding Lagrangian mean that the Kirchhoff current law is not included in the framework yet. This implies that the above EL equation (1)

¹ The class of admissible dissipative elements is restricted to the (rather broad) class that can be described as current-controlled resistors. There are close connections with the notion of *content* as introduced in the early 1950s, see [18] for more details.

is only able to describe circuits with one mesh, except for a parallel connection of a resistor in such circuit. In the latter case, the circuit consists of two meshes, where the additional influence of the mesh that includes the resistor on the dynamical equations is completely determined by a Kirchhoff current law that can be included via the Rayleigh dissipation function.

The framework described above has been used for the EL modeling of the Boost, Buck and Buck–Boost converter in [9,14]. In [9] an attempt is being made to model the Ćuk and Boost–Boost converters (both circuits with more than one mesh) according to this framework. However, the procedure is ad hoc and some of the obtained coordinates do not have a physical meaning. We are interested in a general method for dynamic modeling of a broad class² of electrical networks in the Lagrangian framework, with or without switches. We consider ideal physical elements, and want to include it in the above-mentioned Lagrangian framework, so that the energy storage structure is transparent, and the physics can be used for control purposes. However, as mentioned above, the EL equations correspond with the Kirchhoff voltage laws, where the branch relations are already substituted into the equation. In order to derive the equations, we need both the generalized position and generalized velocity coordinates. Physically, our first guess would be that we only need the charge on capacitive elements, and the current through inductive elements. However, in order to be able to include circuits with parallel branches, we need to incorporate the currents through the branches with the capacitive element so that we can include the Kirchhoff current laws in our framework. Therefore, to build up our framework, we attach to each energy storage element, i.e., inductor or capacitor, two-state variables, namely a charge and a current. Physically, it can be viewed as if for the inductor the charge is an intermediate help variable, and for the capacitor the current is. In order to involve also the Kirchhoff current laws, we need to consider the

constraint form of the EL equations, see e.g. [16], given by

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}) \right) - \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) \\ &= - \frac{\partial \mathcal{D}}{\partial \dot{q}}(\dot{q}) + A(q)\lambda + \mathcal{F}_q, \end{aligned} \quad (3)$$

$$A(q)^T \dot{q} = 0,$$

where λ is the Lagrange multiplier. Eq. (3) represents the general form of the constraint EL equations. For the systems under consideration the constraint equations are given by the Kirchhoff current laws, which implies that $A(q)$ does not depend on the charge q , and thus is a constant $n \times c$ matrix, where c is the number of constraints. Removing the algebraic constraint equation then finally results in the removal of the intermediate help variables. This procedure can be performed for all topologically complete electrical networks with or without ideal switches. It also accounts for an odd number of dynamical elements in an electrical network. In general, at first sight one may be tempted to exclude such electrical networks from modeling in the EL framework, since the equations are presented in second-order form and it is well known that the number of states corresponds to the number of dynamic elements. However, the “intermediate” help variables in combination with the constraint equations deal with this matter. Also, the switches can be naturally involved in the constraints that follow from Kirchhoff’s current laws, i.e., the dependency on the switch may appear in the Rayleigh dissipation function, in the generalized forcing functions and/or in the constraint equations. In case the network contains one or more switches, we denote the switch position(s) with $u = (u_1, \dots, u_m)$ where $u_i \in U_s := \{0, 1\}$, $i = 1, \dots, m$, i.e., ON or OFF, or in other words u is in the discrete set U_s^m . Re-defining the switch positions may also result in $u_i \in \{-1, 0, 1\}$, $i = 1, \dots, m$, depending on the application and preference of the modeler. Following [9] for the unconstraint case, we refer to the set of functions $(\mathcal{T}, \mathcal{V}, \mathcal{D}, \mathcal{F}_q, A)$ as the *EL parameters* of the circuits, and simply express a circuit Σ by means of the five tuple:

$$\Sigma = \{\mathcal{T}(q, \dot{q}), \mathcal{V}(q), \mathcal{D}_u(\dot{q}), \mathcal{F}_{q_u}, A_u\}. \quad (4)$$

² This class covers all topologically complete electrical circuits. A circuit is called ‘topologically complete’ if it can be described by an independent set of inductor currents and capacitor voltages such that Kirchhoff’s laws are satisfied. For a detailed treatment, the reader is referred to [18].

Here, the dependency of the position of the switch is presented by a subscript. The complete procedure is as follows:

Procedure

- (1) *Dynamic variables*: Relate to each dynamic element of the network, $i = 1, \dots, n$, two coordinates, namely a charge and a current coordinate, q_i , and \dot{q}_i , $i = 1, \dots, n$.
- (2) *Energy*: Determine the corresponding energy for all ideal elements, i.e., the magnetic co-energy for the inductive elements, denoted by $\mathcal{T}(q, \dot{q})$, and the electric field energy for the capacitive elements, denoted by $\mathcal{V}(q)$. In case of a switching network, this step does *not* involve the position(s) of the switch(es).
- (3) *Dissipation*: Determine the Rayleigh dissipation function, denoted by $\mathcal{D}(\dot{q}, u)$, for the resistive elements, which may involve the switch position(s) u , and the use of a Kirchhoff current law for determining the current through the resistive element in terms of the dynamic elements as given in step 1.
- (4) *Forcing functions*: Determine the generalized forcing functions $\mathcal{F}_q(u)$ given by the voltage sources, possibly depending on the switch position(s).
- (5) *Constraints and interconnection*: Give the constraint equations that are determined by Kirchhoff's currents laws, that do not include the laws of step 3, and thus only involve the currents through the dynamic elements. If there are no constraint equations for this step, then $A_u = 0$.
- (6) *Equations of motion*: Substitute the information of the previous steps in the constraint form of the EL equations (3) and determine a state space model by choosing the currents corresponding with the inductive elements, and either the charge or the voltage corresponding with the capacitive elements, as state variables.

Remark. In case the circuit contains inductor-only cutsets or capacitor-only loops (excess elements), the circuit is said to be topologically “over”-complete. Consequently, we cannot directly define an independent set of generalized coordinates in such cases.

Application of the constraint equations naturally results in a minimal order description.

Next, we illustrate the procedure by three examples that exhibit interesting properties. First, we study an LC-circuit with an odd number of states, then we study a Čuk converter circuit with an ideal switch, and finally, we study a three-phase boost rectifier with multiple switches. All examples use the constraint EL equations (3). However, the above procedure also applies to the unconstrained case, i.e., where $A = 0$, like in the examples of the Buck, Boost and Buck–Boost converter circuits, see [14], where the EL models of these systems have been obtained on a case by case basis. Throughout the document we explicitly assume that the converters operate in continuous conduction mode.

Example 1 (LC-circuit, Fig. 1). This example is also studied in [15, Example 4.2.1], and illustrates the procedure for an electrical network with an odd number of dynamic elements. (The dot symbol (\bullet) indicates where the current is flowing into the inductor.)

Step 1 of the procedure results in the (intermediate) state variables given by q_i, \dot{q}_i , $i = \{L_1, C_1, L_2\}$, with $q = (q_{L_1}, q_{C_1}, q_{L_2})^T$. The energy in step 2 is given by the Lagrangian

$$\begin{aligned} \mathcal{L}(\dot{q}, q) &= \mathcal{T}(\dot{q}_{L_1}, \dot{q}_{L_2}) - \mathcal{V}(q_{C_1}) \\ &= \frac{1}{2} L_1 \dot{q}_{L_1}^2 + \frac{1}{2} L_2 \dot{q}_{L_2}^2 - \frac{1}{2C_1} q_{C_1}^2. \end{aligned}$$

Since there is no dissipation we set $\mathcal{D}(\dot{q}) = 0$, and we continue with step 4. The forcing functions are given by the voltage source $\mathcal{F}_{q_{L_1}} = E$ and $\mathcal{F}_{q_{C_1}} = \mathcal{F}_{q_{L_2}} = 0$. The constraint equations from step 5 are given by the Kirchhoff current law as follows:

$$\dot{q}_{L_1} - \dot{q}_{C_1} - \dot{q}_{L_2} = 0.$$

Then, with $A^T = [1, -1, -1]$, we obtain from Eq. (3)

$$L_1 \ddot{q}_{L_1} = \lambda + E,$$

$$\frac{1}{C_1} q_{C_1} = -\lambda,$$

$$L_2 \ddot{q}_{L_2} = -\lambda,$$

$$0 = \dot{q}_{L_1} - \dot{q}_{C_1} - \dot{q}_{L_2},$$

which results in the dynamical equations corresponding to (4.19) in [15], with $x = (x_1, x_2, x_3)^T = (\dot{q}_{L_1}, \frac{1}{C_1} q_{C_1}, \dot{q}_{L_2})^T$,

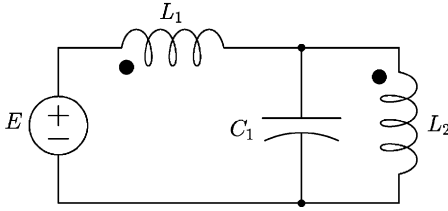


Fig. 1. LC circuit for Example 1.

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{L_1} x_2 + \frac{1}{L_1} E, \\ \dot{x}_2 &= \frac{1}{C_1} x_1 - \frac{1}{C_1} x_3, \\ \dot{x}_3 &= \frac{1}{L_2} x_2.\end{aligned}$$

Example 2 (Ćuk converter, Fig. 2). This example illustrates the potential of the proposed procedure for switching networks. Note that the sign of L_2 in Fig. 2 is opposite to that of the rest of the elements. This makes it easier to include coupled magnetics as is done in Section 3.

Step 1 of the procedure results in the (intermediate) state variables: q_i, \dot{q}_i , $i = L_1, C_1, L_2, C_2$, with $q = (q_{L_1}, q_{C_1}, q_{L_2}, q_{C_2})^T$. The energy in step 2 is given by the Lagrangian

$$\begin{aligned}\mathcal{L}(q, \dot{q}) &= \mathcal{T}(\dot{q}_{L_1}, \dot{q}_{L_2}) - \mathcal{V}(q_{C_1}, q_{C_2}) \\ &= \frac{1}{2} L_1 \dot{q}_{L_1}^2 + \frac{1}{2} L_2 \dot{q}_{L_2}^2 - \frac{1}{2C_1} q_{C_1}^2 - \frac{1}{2C_2} q_{C_2}^2.\end{aligned}$$

The Rayleigh dissipation function of step 3 is given by the free energy dissipated through the load, i.e.,

$$\mathcal{D}(\dot{q}_{L_2}, \dot{q}_{C_2}) = \frac{1}{2} R_1 (-\dot{q}_{L_2} - \dot{q}_{C_2})^2.$$

The forcing functions of step 4 are given by the voltage source, i.e., $\mathcal{F}_{q_{L_1}} = E$, and by $\mathcal{F}_{q_{C_1}} = \mathcal{F}_{q_{L_2}} = \mathcal{F}_{q_{C_2}} = 0$. The constraint equations of step 5 are given by the Kirchhoff current law as follows:

$$\dot{q}_{C_1} + u \dot{q}_{L_2} - (1 - u) \dot{q}_{L_1} = 0$$

and thus $A_u^T = [-(1 - u), 1, u, 0]$. Substitution of the latter into Eqs. (3) yields for the Ćuk converter:

$$L_1 \ddot{q}_{L_1} = -(1 - u) \lambda + E,$$

$$\frac{1}{C_1} q_{C_1} = \lambda,$$

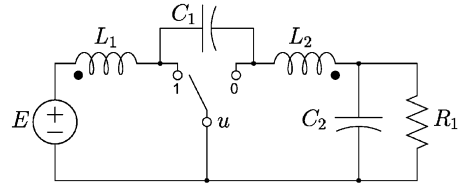


Fig. 2. Ćuk converter circuit for Example 2.

$$L_2 \ddot{q}_{L_2} = R_1 (-\dot{q}_{L_2} - \dot{q}_{C_2}) + u \lambda,$$

$$\frac{1}{C_2} q_{C_2} = R_1 (-\dot{q}_{L_2} - \dot{q}_{C_2}),$$

$$0 = \dot{q}_{C_1} + u \dot{q}_{L_2} - (1 - u) \dot{q}_{L_1},$$

which results with $x = (\dot{q}_{L_1}, \frac{1}{C_1} q_{C_1}, \dot{q}_{L_2}, \frac{1}{C_2} q_{C_2})^T$ in the state equations

$$\begin{aligned}\dot{x}_1 &= -(1 - u) \frac{1}{L_1} x_2 + \frac{E}{L_1}, \\ \dot{x}_2 &= (1 - u) \frac{1}{C_1} x_1 - u \frac{1}{C_1} x_3, \\ \dot{x}_3 &= u \frac{1}{L_2} x_2 + \frac{1}{L_2} x_4, \\ \dot{x}_4 &= -\frac{1}{C_2} x_3 - \frac{1}{R_1 C_2} x_4.\end{aligned}\tag{5}$$

Example 3 (Three-phase boost rectifier, Fig. 3).

This example is meant to illustrate the procedure for obtaining an EL model for systems with multiple switches, and includes constraints on the sources. This example is also studied in [3,19].

Let u_1 , u_2 and u_3 denote the (ideal) switching functions, taking the values 0 or 1. In this case the admissible control vectors are in the discrete set U_s^3 as

$$\begin{aligned}U_s^3 &= [U_1, \dots, U_8] \\ &= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.\end{aligned}\tag{6}$$

The input supply is assumed to be a balanced sinusoidal voltage source, i.e., $e_1 + e_2 + e_3 = 0$, and

$$e_1(t) = E \cos(\omega t),$$

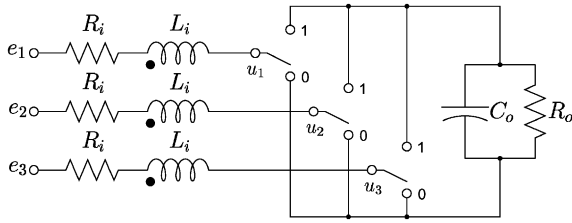


Fig. 3. Three-phase Boost rectifier for Example 3, with line resistances R_i and input inductive elements L_i .

$$e_2(t) = E \cos(\omega t - \phi), \quad (7)$$

$$e_3(t) = E \cos(\omega t + \phi),$$

where $\phi = (2\pi/3)(\text{rad})$ and E is a positive constant. We also assume that there is no neutral line, which results in a Kirchhoff constraint on the currents through the inductive elements $L_k = L_i$, $k = 1, 2, 3$. We then proceed with step 1 of the procedure by defining as the (intermediate) state variables q_j, \dot{q}_j , $j = \{L_k, C_o\}$, for $k = 1, 2, 3$, and thus $q = (q_{L_1}, q_{L_2}, q_{L_3}, q_{C_o})^T$. The energy of step 2 results in the Lagrangian

$$\begin{aligned} \mathcal{L}(q, \dot{q}) &= \mathcal{T}(\dot{q}_{L_k}) - \mathcal{V}(q_{C_o}) \\ &= \frac{1}{2} L_i \sum_{k=1}^3 \dot{q}_{L_k}^2 - \frac{1}{2C_o} q_{C_o}^2. \end{aligned} \quad (8)$$

The Rayleigh dissipation function of step 3 is given by the dissipated free energy over the load and the input resistances, i.e.,

$$\begin{aligned} \mathcal{D}_u(\dot{q}_{L_k}, \dot{q}_{C_o}) &= \frac{1}{2} R_o \left(\sum_{k=1}^3 u_k \dot{q}_{L_k} - \dot{q}_{C_o} \right)^2 \\ &\quad + \frac{1}{2} R_i \sum_{k=1}^3 \dot{q}_{L_k}^2. \end{aligned}$$

The forcing functions of step 4 are given by e_1 , e_2 and e_3 as follows:

$$\mathcal{F}_{q_{L_k}} = e_k, \quad k = 1, 2, 3, \quad \mathcal{F}_{q_{C_o}} = 0.$$

Different from the previous examples, now we have a constraint on the input sources, namely, $\mathcal{F}_{q_{L_1}} + \mathcal{F}_{q_{L_2}} + \mathcal{F}_{q_{L_3}} = 0$. If desired, one of the input sources could easily be eliminated by this constraint. The constraint equation of step 5 is now a result from our assumption that the source has no neutral line, and reads as

$$A^T \dot{q} = \dot{q}_{L_1} + \dot{q}_{L_2} + \dot{q}_{L_3} = 0.$$

Hence, $A^T(q) = [1, 1, 1, 0]$. After substituting the above information into the constraint equation (3), we obtain for the three-phase Boost rectifier:

$$\begin{aligned} L_i \ddot{q}_{L_k} &= -R_i \dot{q}_{L_k} - R_o \left(\sum_{k=1}^3 u_k \dot{q}_{L_k} - \dot{q}_{C_o} \right) u_k + \lambda + e_k, \\ k &= 1, 2, 3, \end{aligned} \quad (9)$$

$$\frac{1}{C_o} q_{C_o} = R_o \left(\sum_{k=1}^3 u_k \dot{q}_{L_k} - \dot{q}_{C_o} \right), \quad (10)$$

$$0 = \dot{q}_{L_1} + \dot{q}_{L_2} + \dot{q}_{L_3}. \quad (11)$$

Hence, λ can be solved from the above equations, and we find

$$\lambda = \frac{1}{3C_o} q_{C_o} \sum_{k=1}^3 u_k. \quad (12)$$

Finally, by letting $x = (\dot{q}_{L_1}, \dot{q}_{L_2}, \dot{q}_{L_3}, \frac{1}{C_o} q_{C_o})^T$, we obtain

$$\dot{x}_1 = \frac{1}{L_i} \left(e_1 - R_i x_1 - \left(u_1 - \frac{1}{3} \sum_{k=1}^3 u_k \right) x_4 \right)$$

$$\dot{x}_2 = \frac{1}{L_i} \left(e_2 - R_i x_2 - \left(u_2 - \frac{1}{3} \sum_{k=1}^3 u_k \right) x_4 \right)$$

$$\dot{x}_3 = \frac{1}{L_i} \left(e_3 - R_i x_3 - \left(u_3 - \frac{1}{3} \sum_{k=1}^3 u_k \right) x_4 \right)$$

$$\dot{x}_4 = \frac{1}{C_o} u_1 x_1 + \frac{1}{C_o} u_2 x_2 + \frac{1}{C_o} u_3 x_3 - \frac{1}{R_o C_o} x_4$$

$$0 = x_1 + x_2 + x_3$$

with $e_1 + e_2 + e_3 = 0$. The algebraic constraint (11) on the states stemming from the Kirchhoff current law has not been eliminated yet. This implies that the above description is actually a non-minimal state space description. A minimal description can be obtained by deleting the dynamic equation for, e.g. x_3 , and substitute $x_3 = -x_1 - x_2$ into the dynamical equation for x_4 . However, the most efficient and useful minimal system description is obtained if the system is transformed by an orthogonal transformation into the so-called $\alpha\beta$ -coordinates (see the remark hereafter). In particular, the $\alpha\beta$ -coordinate transformation becomes of importance if one wants to apply, for example, the

PBC design technique. This is due to fact that the interconnection structure has to satisfy the ‘integrability’ conditions, i.e., has to be skew symmetric, see e.g. [9].

Remark. In three- or multi-phase power electronic networks it is often assumed that the source voltages satisfy certain constraints. For a l -phase network with n dynamic elements and with input voltages that form a balanced source, these constraints are of the form $\mathcal{F}_{q_{i_1}} + \dots + \mathcal{F}_{q_{i_n}} = 0$ where $i_j \in \bar{n} := 1, \dots, n$, $j = 1, \dots, l$, $l \leq n$ and $i_j \neq i_k$ for $j \neq k$. For a symmetrical l -phase network with a balanced source and no neutral line this results in constraints on the currents through the input inductances as well. From a system theoretic point of view the latter implies that the system is non-minimal in the present description. In general there are many ways to deal with this type of algebraic dependence. In the field of electrical machines and power electronic networks, a very often used and convenient method is to transform the system into an orthogonal fixed ($\alpha\beta$) or rotating (dq) reference frame, see e.g. [7]. In case of a three-phase network these transformations are defined as follows. Let $g_1(t) + g_2(t) + g_3(t) = 0$, then in the $\alpha\beta$ and dq coordinate frame g_1, g_2 and g_3 are expressed as

$$g_\alpha + jg_\beta = \sqrt{\frac{2}{3}} (g_1 + g_2 e^{-j\phi} + g_3 e^{j\phi})$$

$$g_d + jg_q = (g_\alpha + g_\beta) e^{-j\psi},$$

where $\phi = 2\pi/3$ and $\psi \in \mathbb{R}$. For sinusoidal functions ψ is usually expressed in terms of the radial frequency denoted by $\omega(t)$, i.e.,

$$\psi(t) = \int_0^t \omega(t') dt'.$$

As shown in [9,12,14], the switched EL equations are also closely related to the average PWM models. A PWM policy regulating the switch position function u may be specified as follows:

$$u(t) = \begin{cases} 1 & \text{for } t_k \leq t < t_k + D(t_k)T, \\ 0 & \text{for } t_k + D(t_k)T \leq t < t_k + T, \end{cases}$$

$$t_{k+1} = t_k + T, \quad k = 0, 1, \dots, \quad (13)$$

where t_k represents a sampling instant; the parameter T is the fixed sampling period; the sampled value of the state vector $x(t)$ of the converter are denoted by

$x(t_k)$. The function $D(\cdot)$ is the *duty ratio function*, acting as an external control input to the average PWM model of the converter [13]. The value of the duty ratio function $D(t_k)$ determines at every sampling instant t_k the width of the upcoming ON pulse as $D(t_k)T$ (during this period the switch is fixed at the position represented by $u = 1$). Now, the duty ratio function $D(\cdot)$ is evidently a function limited to take real values on the open interval $\langle 0, 1 \rangle$.

For networks with a switch, note that according to the PWM switching policy (13) at every sampling interval of period T , the Kirchhoff constraint $A^T \dot{q} = 0$ for $u = 1$ is valid over only a fraction of the sampling period given by $D(t_k)$, while the constraint for $u = 0$ is valid over only a fraction of the sampling period equal to $(1 - D(t_k))$. One possible way to handle this, is to consider an average value of the Kirchhoff constraints, and thus to present the set of EL parameters as dependent on the duty cycle in the same way as in the original procedure is dependent on the discrete values of the switch. Note that if we would take $D = 1$ or 0, one recovers, respectively, the Kirchhoff constraint for the two switch positions. We note that the Lagrangian function associated with the above defined average EL parameters did not change with respect to the switch position function.

Example 2 (Continued). The PWM model for the Ćuk converter is given by the dynamic equations as in (5), where the state variable x is replaced by the average state z , i.e., z_1 and z_3 represent the average inductor currents, z_2 and z_4 the average capacitor voltages, and where the discrete signal u is replaced by the duty cycle $D(t)$ that takes values in the open interval $\langle 0, 1 \rangle$.

Remark. The presented EL modeling technique for (switching) networks results in the same dynamical models as when the Hamiltonian framework is used, e.g. [8,5], provided that the same level of ideal physics is assumed. However, the Hamiltonian framework does not introduce the ‘semi’-physical intermediate help variables. Nevertheless, we do emphasize that the above framework is an easy, general, and straightforward way to obtain the dynamic models of electrical networks, where the interconnection between the elements, given by the Kirchhoff laws, corresponds to the straightforward knowledge of the electrical engineer. It gives us the opportunity

to apply the well-known PBC techniques, as presented in [14] for the Buck, Boost and Buck–Boost converter, for general switching electrical network structures.

Remark. The well-known classical mesh current analysis method (e.g. [2]) results in the same dynamical behavior. Based on the Kirchhoff current laws, the mesh currents are taken as a starting point to determine the dynamical equations, i.e., they automatically satisfy the Kirchhoff current laws. Then, the Kirchhoff voltage law for each mesh is considered, resulting in the basic equations, where the dynamical relations given by the branches should be substituted in. Our framework takes as a starting point the EL equations that correspond to the Kirchhoff voltage laws, which seems to be more related to the classical node voltage analysis (e.g. [2]). A limitation of mesh current analysis is that it is only applicable to planar networks, where as our framework and also node voltage analysis does not have this limitation. However, the main difference of both mesh current and node voltage analysis with our framework is given by the fact that they do not take energy storage and passivity as the characteristics that should be explicitly used and present in the model for analysis and control purposes.

Remark. The proposed modeling procedure and presentation bears some similarities with the approach from a behavioral point of view [10]. The *manifest* variables that can completely describe the dynamical behavior of the electrical circuits are given by the current through the inductors, \dot{q}_L , the charge on the capacitors, q_C , the source voltage, and the switch position. The so-called intermediate help variables given by the branch current of the capacitor, \dot{q}_C , and the charge on the inductor, q_L , can be viewed as *latent* variables in the behavioral setting. However, other choices of manifest and latent variables can easily be made. For example, when we are interested in the control of the output voltage via the switch position, the manifest variables are given by $C^{-1}q_C$ and u , while all others are latent variables. The latter choice corresponds to an input/output view as is often considered for electrical circuits. Also in our perspective, the EL modeling procedure is partly motivated by the design and application of PBC for switching power

converters, where the purpose is to control the voltage over the load (output voltage) via the switch position (input).

3. Coupled-magnetics

In this section, we treat the inclusion of coupled-magnetics in the EL framework. Though magnetic coupling of inductors is well known from circuit theory, e.g. [1], we consider it for inclusion in the EL framework. As an example, we illustrate the potential of the method using the Ćuk converter of Example 2, in which both the inductors are coupled, e.g. [4].

A pair of coupled-inductors may be considered as the non-ideal equivalent of a transformer, with a rate of coupling, $k \in [0, 1)$ and an effective turns ratio

$$n = \sqrt{\frac{L_1}{L_2}}.$$

As a result of the coupling, both the magnetizing currents share the same flux paths with an order or magnitude depending on k . This involves an additional path for the energy using a magnetic field. In terms of the common fluxes ϕ_j , $j=1, 2$, a pair of coupled-inductors can be characterized as follows:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \underbrace{\begin{bmatrix} L_1 & L_{12} \\ L_{21} & L_2 \end{bmatrix}}_{=: M} \begin{bmatrix} i_{L_1} \\ i_{L_2} \end{bmatrix} \quad (14)$$

for which $L_{12}, L_{21} \geq 0$ satisfy the reciprocity condition, i.e., $L_{12} = L_{21} = L_m$, where $L_m = k\sqrt{L_1 L_2}$ is denoted as the mutual inductance.

From these relations it is clear that L_1 and L_2 would form two separate inductors if $k = 0$, and thus $L_m = 0$. The differential equation relating the currents and voltages for the block Σ_M in Fig. 4 is obtained from (14) by applying Faraday's law, i.e., $v_{L_j} = d\phi_j/dt$, $j = 1, 2$, as

$$\Sigma_M = \frac{d}{dt} \begin{bmatrix} i_{L_1} \\ i_{L_2} \end{bmatrix} = \begin{bmatrix} \beta & -\gamma \\ -\gamma & \alpha \end{bmatrix} \begin{bmatrix} v_{L_1} \\ v_{L_2} \end{bmatrix}, \quad (15)$$

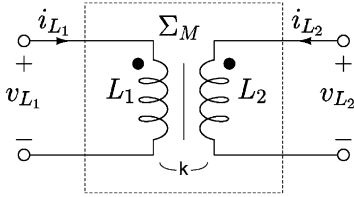


Fig. 4. The block representing two coupled inductors.

where α , β and γ are given by

$$\alpha = \frac{n^2}{(1-k^2)L_1}, \quad \beta = \frac{1}{(1-k^2)L_1},$$

$$\gamma = \frac{nk}{(1-k^2)L_1}. \quad (16)$$

The term $(1-k^2)$ can be considered as the magnetic flux dispersal, which denotes the amount of flux not shared by both the inductors. Notice that other parameterizations of (16) are also possible but, as will be illustrated later in the example, these notations provide a straightforward insight in the magnetizing energy interconnections. In view of the Lagrangian modeling procedure, we consider a pair of magnetically coupled inductors that satisfy the reciprocity condition as a single system Σ_M for which the total amount of stored energy is given by the kinetic co-energy $\mathcal{T}(\dot{q}_L) = \frac{1}{2} \dot{q}_L^T M \dot{q}_L$, with $L_{12} = L_{21} = L_m$. Since $0 \leq k < 1$, $\mathcal{T}(\dot{q}_L) > 0$ for all $\dot{q}_L \neq 0$. The extension to the magnetic energy of an electrical network containing N inductors with N magnetically coupled loops follows straightforwardly along the same lines.

Example 2 (Continued). *coupled-inductor Ćuk converter.* In this example we include the coupled magnetics for the Ćuk converter in the EL modeling procedure.

The circuit topology of the coupled-inductor Ćuk converter is given in Fig. 5. The capacitive energy transfer imposes identical rectangular voltage waveforms on both the inductors which has justified the magnetic extension [4]. Providing the right adjustments of k and n , the input current ripple can be steered into the output inductor, or vice versa, to result in practically zero ripple current on either the input or the output of the converter. In Example 2 the general procedure of Section 2 has been followed. This can still be done in the coupled inductor case, where the only

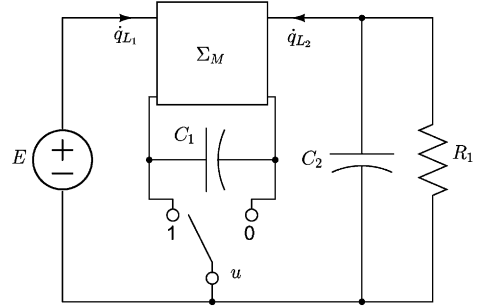


Fig. 5. Coupled-inductor Ćuk converter.

difference can be found in the magnetic co-energy, i.e., the magnetic co-energy, $\mathcal{T}(q, \dot{q})$, of the circuit is given by

$$\mathcal{T}(\dot{q}_{L_1}, \dot{q}_{L_2}) = \frac{1}{2} L_1 \dot{q}_{L_1}^2 + \frac{1}{2} L_2 \dot{q}_{L_2}^2 + L_m \dot{q}_{L_1} \dot{q}_{L_2}. \quad (17)$$

The remaining EL parameters stay the same. Applying the constraint EL equations with the same states as in Example 2, results in terms of α , β and γ in the following state space model:

$$\begin{aligned} \dot{x}_1 &= -u\gamma x_2 - (1-u)\beta x_2 - \gamma x_4 + \beta E, \\ \dot{x}_2 &= (1-u)\frac{1}{C_1} x_1 - u\frac{1}{C_1} x_3, \\ \dot{x}_3 &= u\alpha x_2 + (1-u)\gamma x_2 + \alpha x_4 - \gamma E, \\ \dot{x}_4 &= -\frac{1}{C_2} x_3 - \frac{1}{R_1 C_2} x_4. \end{aligned} \quad (18)$$

This is the model with discrete values for the switch, where x_1, x_3 represent the inductor currents, and x_2, x_4 represent the capacitor voltages. From the definitions in (16) it is now easy to see that the relation between the effective turns ratio n and the rate of coupling k affects the system properties, and in particular the influence of the switching effect in the current equations. For the so-called *matching condition*, where $n = k$, and thus $\alpha = \gamma$, the state x_3 does not depend on the switch position function anymore, which results in zero ripple output current x_3 . The same holds for x_1 in case of the so-called *inverse matching condition*, $n = k^{-1}$, $\beta = \gamma$, which results in zero ripple input current x_1 . A third relevant practical condition can be found for $n = 1, 0 \leq k < 1$, $\alpha = \beta$, which is the so-called *balanced ripple reduction condition*. In that case both the input and output current ripples can be reduced

(relative to the situation where there is no magnetic coupling) up to 50%, depending on the value of k .

Remark. The matching, inverse matching and balanced ripple conditions, unfortunately, are often not easy to acquire in practice, i.e., there will always be a certain ‘mismatch’ between k and n . Coupled-inductor extensions can also be applied to the classical Buck, Boost and Buck–Boost converters, but in these cases an extra capacitor is needed to serve as a driven voltage source for the secondary inductor, see e.g. [17].

4. Conclusions

We have presented a systematic procedure for Lagrangian modeling of (switching) electrical networks. Modeling within the EL framework yields the opportunity to explicitly model the system based on the energy, so that passivity properties are incorporated in a transparent manner in the model and thus physical controller design methods can be easily applied. So far, some electrical networks have been modeled within the Lagrangian framework, but there was no systematic way to do so for more general types of networks. Here we have included ideal dynamical elements, dissipative elements, switches and magnetic coupling of inductive elements into the framework.

Application of the physical control design techniques to networks with realistic switching frequencies can be found in [6,11]. A thorough design analysis and physical tuning guideline are given in [6] for the damping injection of such controllers.

Future research includes the study of the interconnection structure of these type of networks and of the relation with the Hamiltonian framework and interconnection structure.

References

- [1] N. Balabanian, Fundamentals of Circuit Theory, Allyn and Bacon, Boston, 1961.
- [2] N. Balabanian, T.A. Bickart, S. Seshu, Electrical Network Theory, Wiley, New York, 1969.
- [3] V. Blasko, V. Kaura, A new mathematical model and control of a three-phase AC–DC voltage source converter, IEEE Trans. Power Elec. 12 (1997) 116–123.
- [4] S. Čuk, R.D. Middlebrook, Coupled-inductor and other extensions of a new optimum topology switching DC-to-DC converter, Advances in Switched-Mode Power Conversion, Vol. 2, TESLAco, Pasadena, CA, 1981, pp. 331–347.
- [5] G. Escobar, A.J. van der Schaft, R. Ortega, A Hamiltonian viewpoint in the modeling of switching power converters, Automatica 35 (1999) 445–452.
- [6] D. Jeltsema, J.M.A. Scherpen, J.B. Klaassens, Energy-control of multi-switch power supplies: an application to the three-phase Buck rectifier with input filter, Proceedings of the IEEE Power Electronics Specialists Conference 2001, No. 26–7, Vancouver, Canada, June 2001.
- [7] W. Leonhard, Control of Electrical Drives, Elsevier, Amsterdam, 1985.
- [8] B. Maschke, R. Ortega, A.J. van der Schaft, Energy-based Lyapunov functions for forced Hamiltonian systems with dissipation, IEEE Trans. Automat. Control AC-45 (2000) 1498–1502.
- [9] R. Ortega, A. Loria, P.J. Nicklasson, H. Sira-Ramírez, Passivity Based Control of Euler–Lagrange Systems: Mechanical, Electrical and Electromechanical Applications, Springer, London, 1998.
- [10] J.W. Polderman, J.C. Willems, Introduction to Mathematical Systems Theory: A Behavioral Approach, Springer, New York, 1998.
- [11] J.M.A. Scherpen, D. Jeltsema, J.B. Klaassens, Lagrangian modeling and control of switching networks with integrated coupled magnetics, Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, Australia, December 2000, pp. 4054–4059.
- [12] J.M.A. Scherpen, J.B. Klaassens, L. Ballini, Lagrangian modeling and control of DC-to-DC converters, Proceedings of the IEEE INTELEC, Copenhagen, 99CH37007, No. 31–41, June 1999.
- [13] H. Sira-Ramírez, A geometric approach to pulse-width-modulated control in nonlinear dynamical system, IEEE Trans. Automat. Control AC-34 (1989) 184–187.
- [14] H. Sira-Ramírez, R.A. Perez-Moreno, R. Ortega, M. Garcia-Esteban, Passivity-based controllers for the stabilization of DC-to-DC power converters, Automatica 33 (4) (1997) 499–513.
- [15] S. Stramigioli, Modeling and IPC Control of Interactive, Mechanical Systems: A Coordinate Free Approach, Lecture Notes in Control and Information Science, Vol. 266, Springer, London, 2000.
- [16] A.J. van der Schaft, \mathcal{L}_2 -Gain and Passivity Techniques in Nonlinear Control, Springer, London, 2000.
- [17] J. Wang, W.G. Dunford, K. Maunch, Analysis of a ripple-free input-current boost converter with discontinuous conduction characteristics, IEEE Power Electron. 12 (4) (1997) 684–694.
- [18] L. Weiss, W. Mathis, L. Trajkovic, A generalization of Brayton–Moser’s mixed potential function, IEEE Trans. CAS, Fund. Theory Appl. 45 (1998) 423–427.
- [19] R. Wu, S.B. Dewan, G.R. Slemon, A PWM AC-to-DC converter with fixed switching frequency, IEEE Trans. Ind. Appl. 26 (1990) 880–885.